We have received a solution to Problem 5 of the 2000 Russian Mathematical Olympiad [2002 : 483] which avoids the calculus used in the featured solution [2005 : 95–96].

5. Prove that

$$\frac{1}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+y^2}} \;\le\; \frac{2}{\sqrt{1+xy}} \qquad \text{for } 0 < x,\, y \le 1.$$

Solution by Arkady Alt, San Jose, CA, USA, modified by the editor.

We prove the inequality under the slightly weaker conditions that $x,\, y \ge 0$ and $xy \le 1$.

For $x,\, y \ge 0$, let

$$F(x,y) \;=\; \sqrt{1+xy}\left(\frac{2}{\sqrt{1+xy}}-\frac{1}{\sqrt{1+x^2}}-\frac{1}{\sqrt{1+y^2}}\right).$$

Then

$$\begin{aligned} F(x,y) &= \frac{\sqrt{1+x^2}-\sqrt{1+xy}}{\sqrt{1+x^2}}+\frac{\sqrt{1+y^2}-\sqrt{1+xy}}{\sqrt{1+y^2}} \\ &= \frac{x(x-y)}{1+x^2+\sqrt{1+xy}\sqrt{1+x^2}}+\frac{y(y-x)}{1+y^2+\sqrt{1+xy}\sqrt{1+y^2}} \\ &= \frac{(x-y)G(x,y)}{\left(1+x^2+\sqrt{1+xy}\sqrt{1+x^2}\right)\left(1+y^2+\sqrt{1+xy}\sqrt{1+y^2}\right)}, \end{aligned}$$

where

$$\begin{aligned} G(x,y) &= x\left(1+y^2+\sqrt{1+xy}\sqrt{1+y^2}\right) \\ &\qquad -y\left(1+x^2+\sqrt{1+xy}\sqrt{1+x^2}\right) \\ &= x-y-xy(x-y)+\sqrt{1+xy}\left(x\sqrt{1+y^2}-y\sqrt{1+x^2}\right) \\ &= x-y-xy(x-y)+\frac{\sqrt{1+xy}(x^2-y^2)}{x\sqrt{1+y^2}+y\sqrt{1+x^2}} \\ &= (x-y)\left(1-xy+\frac{\sqrt{1+xy}(x+y)}{x\sqrt{1+y^2}+y\sqrt{1+x^2}}\right). \end{aligned}$$

Thus,

$$F(x,y) \;=\; \frac{(x-y)^2\left(1-xy+\dfrac{\sqrt{1+xy}(x+y)}{x\sqrt{1+y^2}+y\sqrt{1+x^2}}\right)}{\left(1+x^2+\sqrt{1+xy}\sqrt{1+x^2}\right)\left(1+y^2+\sqrt{1+xy}\sqrt{1+y^2}\right)}.$$

It is now evident that $F(x,y) \ge 0$ if $xy \le 1$. The desired result follows.